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Elastoplastic material with isotropic damage

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Abstract

The two damage parameters needed to describe isotropic damage are shown to define a damage domain and the evolution of damage is defined by a path within that domain. Physical limitations on the path to full damage are developed. A thermodynamically consistent set of constitutive equations with which the damage path can be evaluated is developed for a specific material.

1. Introduction

In Cauvin and Testa (1997) we have shown that in the most general case, damage that causes stiffness degradation can be described by an eighth-order tensor $\mathbf{D_8}$. If the principle of strain equivalence is imposed (Lemaître and Chaboche, 1990) a tensor of order four, $\mathbf{D}$, is all that is needed to relate damaged and undamaged moduli,

$$\tilde{E}_{ijkl} = (I_{ijmn} - D_{ijmn}) E_{mnkl} \tag{1.1}$$

with

$$I_{ijmn} = \tfrac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{1.2}$$

In the special case of isotropic damage that takes an isotropic undamaged material to an isotropic damaged state, only two elements of the damage tensor $\mathbf{D}$ are independent so that

$$D_{ijkl} = D_2\delta_{ij}\delta_{kl} + \tfrac{1}{2}(D_1 - D_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.3}$$

In Cauvin and Testa (1997) we have also expressed isotropic damage in terms of physically meaningful damage parameters ($D_E, D_S, D_K, D_v \ldots$) that act directly on the familiar moduli ($E, \mu, K, v \ldots$).

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2. The damage domain

Because there are two scalar damage variables, one can think of a two-dimensional damage space in which the evolution of damage follows some path in (D_1, D_2). This will be limited by the obvious restrictions on the physical stiffnesses $\tilde{K}$ and $\tilde{\mu}$. Therefore,

$$(0 \leqslant \tilde{K} \leqslant K \Leftrightarrow 0 \leqslant D_K \leqslant 1) \Rightarrow 0 \leqslant D_1 + 2D_2 \leqslant 1 \tag{2.1}$$

$$(0 \leqslant \tilde{\mu} \leqslant \mu \Leftrightarrow 0 \leqslant D_S \leqslant 1) \Rightarrow 0 \leqslant D_1 - D_2 \leqslant 1 \tag{2.2}$$

which define the damage domain shown in Fig. 1 in the space D_1, D_2.

For isotropic damage, the evolution of D_1 and D_2 must be such that the damage state remains inside the damage domain which can be restated from eqns (2.1) and (2.2) as:

$$0 \leqslant D_1 \leqslant 1 \tag{2.3}$$

$$-\tfrac{1}{3} \leqslant D_2 \leqslant \tfrac{1}{3} \tag{2.4}$$

The portion of this domain corresponding to negative values of $\tilde{\nu}$ may be excluded from consideration. Therefore, the boundary of the damage domain may be shifted to the line representing $\tilde{\nu} = 0$ which gives the damage domain OABC in Fig. 1 and leads to the limit

$$-\frac{1}{3} \leqslant D_2 \leqslant \frac{\nu}{1+\nu} \tag{2.5}$$

Within this damage domain, it may also be verified that the other physically significant damaged material constants satisfy the following expected limits:

$$0 \leqslant \tilde{E} \leqslant E \tag{2.6}$$

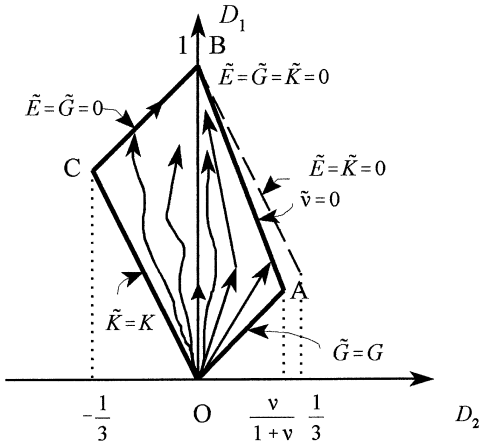

Fig. 1. Damage domain and damage path.

$$0 \leqslant \tilde{\nu} \leqslant \tfrac{1}{2} \tag{2.7}$$

3. Full damage condition

Because the two scalar isotropic damage parameters D_1 and D_2 do not have simple physical meanings, the values that will occur at full damage at a point in the material are not readily apparent. This is so, despite the limits on these parameters, eqns (2.3) and (2.5), which define the damage domain. In other words, in the damage domain of Fig. 1, it is not obvious where full damage occurs.

On the other hand, one can see on physical grounds that full damage must have occurred at a point in the material if it has fully lost any one of the stiffnesses, E, μ, or K. Full damage at a point in the material corresponds, therefore, to any one of the following:

$$D_E = 1, \quad D_S = 1, \quad D_K = 1 \tag{3.1}$$

In the present work we do not account for possible critical values of damage that precipitate failure before complete loss of stiffness is reached.

4. Damage paths

Whatever the progression of damage, it will be represented by some path in the space of D_1D_2. Any limitations on the nature of that path must come from thermodynamics and specific material limits. It is known, however, that the path begins at $D_1 = D_2 = 0$, is fully contained in the damage domain of Fig. 1 that assures non-negative moduli, and it must terminate at the condition of full damage. Full damage according to eqn (3.1) occurs only at point B of the damage domain and, therefore, all damage paths tend toward that point. The degenerate case along the path CB of Fig. 1 may also represent full damage as discussed below.

The expression for $\tilde{\nu}$ as given in Cauvin and Testa (1997) can be seen to represent a straight line in the damage domain whenever the damaged Poisson's ratio $\tilde{\nu}$ remains constant, Fig. 2. These are damage paths with constant Poisson's ratio.

It is noted that Poisson's ratio decreases from the undamaged state ($\tilde{\nu} < \nu$) when the damage parameter D_2 is positive and increases ($\tilde{\nu} > \nu$) when D_2 is negative. If D_2 remains zero throughout the damage process, then Poisson's ratio remains unchanged ($\tilde{\nu} = \nu$) which is, thus, an immutable consequence of using only one damage parameter to describe isotropic damage. Nevertheless, it is seen that even in that case, the terminus for full damage is the point B in Fig. 1 as it would be for any other damage path. Usually it will be found that $\tilde{\nu}$ decreases with damage, but the possibility of increasing $\tilde{\nu}$ appears to be borne out by Budiansky and O'Connell (1976). A path on the boundary AB represents damage progressing with $\tilde{\nu} = 0$ and

$$D_2 = \nu(1 - D_1) \tag{4.1}$$

A path on the boundary OA, where $D_1 = D_2$ implies no change in the shear modulus ($\tilde{\mu} = \mu$), whereas on OC, it is K that remains unaffected by damage. A peculiar case occurs on BC where

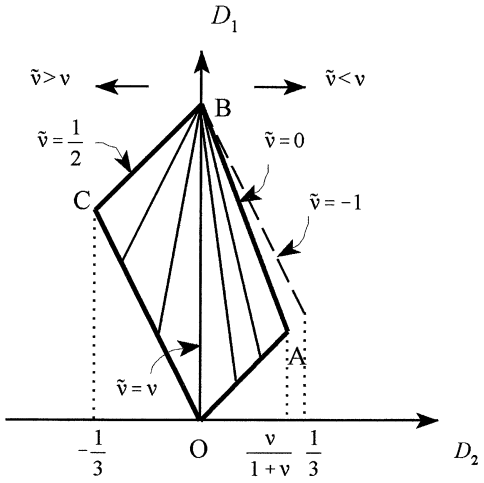

Fig. 2. Lines of constant Poisson's ratio.

$$D_2 = -(1-D_1) \tag{4.2}$$

For points on this boundary of the damage domain we find

$$\tilde{\nu} = \tfrac{1}{2} \tag{4.3}$$

$$\tilde{\lambda} = \frac{E}{1-2\nu}(1-D_1) \tag{4.4}$$

$$\tilde{\mu} = \tilde{E} = 0 \tag{4.5}$$

$$\tilde{K} = \frac{E}{1-2\nu}(1-D_1) \tag{4.6}$$

The damaged material in this state is no longer capable of sustaining shear or uniaxial stresses ($\tilde{E} = \tilde{\mu} = 0$) but can sustain a hydrostatic stress ($\tilde{K} \neq 0$). In this sense, it is technically not fully damaged but can act as an inviscid fluid.

Clearly, it is only for special damage paths that the damage description is reduced to only one parameter, i.e. the often encountered $(1-D)$ characterization of damage (Lemaître and Chaboche, 1990). For example, the case with $D_2 = 0$ that is often considered corresponds to constant Poisson's ratio, $\tilde{\nu} = \nu$, and gives damaged parameters $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{E}$ and $\tilde{K}$ that are simply $(1-D_1)$ times the undamaged values.

In general, however, for isotropic damage the physical moduli are more complicated functions of the two damage parameters D_1 and D_2, as detailed in Cauvin and Testa (1997). From these results, D_1 and D_2 can be evaluated from measurements of the physical moduli using

$$D_1 = 1 - \frac{\tilde{E}[1-\tilde{\nu}(1+2\nu)]}{E(1+\tilde{\nu})(1-2\tilde{\nu})(\nu-\tilde{\nu})} \tag{4.7}$$

$$D_2 = \frac{\tilde{E}}{E(1+\tilde{\nu})(1-2\tilde{\nu})} \tag{4.8}$$

The measurement of Young's modulus and Poisson's ratio in an isotropically damaged material permits evaluation of the two independent damage variables.

The determination of the path along which the damage progresses in the damage space D_1D_2 requires development of constitutive equations for the material. Because the kinetics are not unlike those of plasticity, and because it is likely that in many applications damage will also be accompanied by plastic deformation, they are considered together in the ensuing development.

Isotropic damage models currently found in the literature may all be classified as the $(1-D)$ type for which the degradation of the mechanical properties of materials is assumed to be entirely described by a single parameter (Lemaître and Chaboche, 1990). The limitations of such an assumption have yet to be fully determined, but clearly it will at least lead to some inconsistencies when the actual damage path does not satisfy the condition of constant ν which a single damage parameter implies.

In the following we develop within the framework of the thermodynamics of irreversible processes an isotropic damage model based on the tensor **D** with two independent parameters for a material in which deviations from linear elasticity occur as a result of both damage and plasticity. However, the material is time-independent and suffers only small deformations.

5. Damage operator

It is convenient to work with the inverse of the tensor in eqn (1.1), to be called the damage operator and denoted by Λ,

$$\Lambda_{ijkl} = (I_{ijkl} - D_{ijkl})^{-1} \tag{5.1}$$

Using the damage tensor D_{ijkl} for isotropic damage as given by eqn (1.3) we can show that

$$\Lambda_{ijkl} = \frac{2D_2\delta_{ij}\delta_{kl} + (1-D_1-2D_2)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})}{2(1-D_1+D_2)(1-D_1-2D_2)} \tag{5.2}$$

The effective stress tensor can then be written as

$$\tilde{\sigma}_{ij} = \Lambda_{ijkl}\sigma_{kl} \tag{5.3}$$

The familiar relation of the so-called $(1-D)$ type of isotropic damage can be easily recovered by setting $D_2 = 0$ in eqns (5.2) and (5.3) becomes in this case

$$\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1-D_1} \tag{5.4}$$

6. Thermodynamics variables

The state potential from which the state laws are derived is taken as the Helmholtz free energy in the form (Lemaître and Chaboche, 1990; Lubliner, 1990)

$$\psi = \psi_e + \psi_p \tag{6.1}$$

where ψ_e and ψ_p denote, respectively, the elastic part and the plastic part of the free energy. As usual, this is a specific energy expressed per unit mass.

The total strain may also be decomposed into the elastic and plastic strains

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{6.2}$$

Damage in the present context is directly associated with elastic processes and therefore, we may write

$$\psi_e = \psi_e(\boldsymbol{\varepsilon}^e, \mathbf{D}, T) \tag{6.3}$$

where T is the absolute temperature. The thermodynamic variables corresponding to the elastic strain tensor $\boldsymbol{\varepsilon}^e$ and the damage tensor $\mathbf{D}$ are (using ρ as the mass density)

$$\sigma_{ij} = \rho \frac{\partial \psi_e}{\partial \varepsilon^e_{ij}} \tag{6.4}$$

$$\bar{Y}_{ijkl} = \rho \frac{\partial \psi_e}{\partial D_{ijkl}} \tag{6.5}$$

The latter turns out to have the same physical significance as the strain energy release rate of fracture mechanics but applicable here to damage at the mesoscale.

For isothermal processes, the elastic part of the free energy is given in terms of the elastic part of the strain as

$$\psi_e = \frac{1}{2\rho} \tilde{E}_{mnrs} \varepsilon^e_{mn} \varepsilon^e_{rs} \tag{6.6}$$

Or, using eqn (1.1)

$$\psi_e = \frac{1}{2\rho} (I_{mnpq} - D_{mnpq}) E_{pqrs} \varepsilon^e_{mn} \varepsilon^e_{rs} \tag{6.7}$$

Therefore, the stress tensor $\boldsymbol{\sigma}$, eqn (6.4), may be shown to be (see Fig. 3)

$$\sigma_{ij} = (I_{ijpq} - D_{ijpq}) E_{pqmn} \varepsilon^e_{mn} \tag{6.8}$$

from which

$$\varepsilon^e_{ij} = C_{ijmn} \Lambda_{mnkl} \sigma_{kl} \tag{6.9}$$

where $\mathbf{C}$ is the compliance tensor of the virgin material.

The thermodynamic conjugate of $\mathbf{D}$, eqn (6.5), becomes

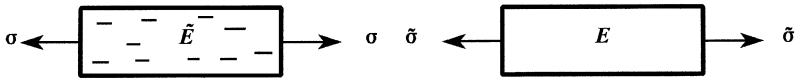

Fig. 3. Stress tensors.

$$\bar{Y}_{ijkl} = -\tfrac{1}{2} E_{klrs}\varepsilon^e_{ij}\varepsilon^e_{rs} = -Y_{ijkl} \tag{6.10}$$

where we have defined the more convenient Y as the negative of $\bar{Y}$.

The plastic part ψ^0_p of the free energy of the undamaged (hence superscript 0) material will be taken in the form

$$\psi^0_p = \psi^0_p(r, \boldsymbol{\alpha}, T) \tag{6.11}$$

where $\boldsymbol{\alpha}$ is the back strain tensor to account for kinematic hardening (Chaboche, 1977) and r is the equivalent plastic strain defined in terms of increments or rates (Odqvist, 1933) as

$$\dot{r} = (\tfrac{2}{3}\dot{\varepsilon}^{p_0}_{ij}\dot{\varepsilon}^{p_0}_{ij})^{1/2} \tag{6.12}$$

where $\varepsilon^{p_0}_{ij}$ is the plastic strain in the undamaged material when it is deformed to the same elastic strain or equivalently when it is subjected to the effective stress $\tilde{\boldsymbol{\sigma}}$. In this form, $\dot{r}$ is just equal to $|\dot{\varepsilon}^{p_0}|$ if a specimen of the undamaged material is subjected to a uniaxial tension or compression.

The thermodynamic variables corresponding to r and $\boldsymbol{\alpha}$ may be written

$$R = \rho\frac{\partial\psi^0_p}{\partial r} \tag{6.13}$$

$$X^D_{ij} = \rho\frac{\partial\psi^0_p}{\partial\boldsymbol{\alpha}_{ij}} \tag{6.14}$$

Here, R is identified as the isotropic strain hardening variable and $\mathbf{X}^D$ the deviatoric back stress tensor (Lemaître and Chaboche, 1990; Lubliner, 1990).

7. Strain energy density

The strain energy density is defined by its increment

$$\mathrm{d}w_e = \sigma_{ij}\,\mathrm{d}\varepsilon^e_{ij} \tag{7.1}$$

in which the elastic strain is the value in the damaged condition and contains both strains due to the initial flexibility and added strain due to damage. At any damage state the stress is linearly related to the total elastic strain and eqn (7.1) may be integrated to obtain

$$w_e = \int\sigma_{ij}\,\mathrm{d}\varepsilon^e_{ij} = \tfrac{1}{2}\sigma_{ij}\varepsilon^e_{ij} \tag{7.2}$$

Introducing eqn (6.8), the strain energy density at constant damage becomes

$$w_e = \tfrac{1}{2}(I_{ijmn} - D_{ijmn})E_{mnkl}\varepsilon^e_{kl}\varepsilon^e_{ij} \tag{7.3}$$

or, using eqn (6.10)

$$w_e = Y_{ijmn}(I_{ijmn} - D_{ijmn}) \tag{7.4}$$

If one considers an increment in damage $\mathrm{d}D_{ijkl}$ at constant stress or strain from some damage state, then the respective increments in strain energy would be

$$\mathrm{d}w_e|_\sigma = \tfrac{1}{2}\sigma_{kl}\,\mathrm{d}\varepsilon^e_{kl}; \quad \mathrm{d}\sigma_{kl} = 0 \tag{7.5}$$

$$\mathrm{d}w_e|_{\varepsilon^e} = \tfrac{1}{2}\varepsilon^e_{kl}\,\mathrm{d}\sigma_{kl}; \quad \mathrm{d}\varepsilon^e_{kl} = 0 \tag{7.6}$$

From eqn (6.8), with $\mathrm{d}\sigma_{ij} = 0$, we find

$$(I_{ijmn} - D_{ijmn})E_{mnkl}\,\mathrm{d}\varepsilon^e_{kl} = \mathrm{d}D_{ijmn}E_{mnkl}\varepsilon^e_{kl} \tag{7.7}$$

Since $\tilde{E}_{ijkl} = \tilde{E}_{klij}$ eqn (7.7) can be written

$$(I_{klmn} - D_{klmn})E_{mnij}\,\mathrm{d}\varepsilon^e_{kl} = \mathrm{d}D_{ijmn}E_{mnkl}\varepsilon^e_{kl} \tag{7.8}$$

If eqn (6.8) is used to replace σ_{kl} in eqn (7.5) then eqn (7.8) will permit reduction to the form

$$\mathrm{d}w_e|_\sigma = \tfrac{1}{2}\mathrm{d}D_{ijmn}E_{mnkl}\varepsilon^e_{kl}\varepsilon^e_{ij} \tag{7.9}$$

Similarly, if we use again eqn (6.8), but with $\mathrm{d}\varepsilon^e_{ij} = 0$, we obtain

$$\mathrm{d}\sigma_{ij} = -\mathrm{d}D_{ijmn}E_{mnkl}\varepsilon^e_{kl} \tag{7.10}$$

which substituted into eqn (7.6) gives

$$\mathrm{d}w_e|_{\varepsilon^e} = -\tfrac{1}{2}\mathrm{d}D_{ijmn}E_{mnkl}\varepsilon^e_{kl}\varepsilon^e_{ij} \tag{7.11}$$

The strain energy increments in eqns (7.9) and (7.11) can be restated by using eqn (6.10) in the form

$$\mathrm{d}w_e|_\sigma = Y_{ijmn}\,\mathrm{d}D_{ijmn} \tag{7.12}$$

$$\mathrm{d}w_e|_{\varepsilon^e} = -Y_{ijmn}\,\mathrm{d}D_{ijmn} \tag{7.13}$$

We see from these that Y_{ijkl} plays the same role as the strain energy release rate of fracture mechanics. It is, therefore, the damage strain energy release rate and in both cases in eqns (7.12) and (7.13) represents the increment in dissipation for a unit increment in damage (Figs 4 and 5).

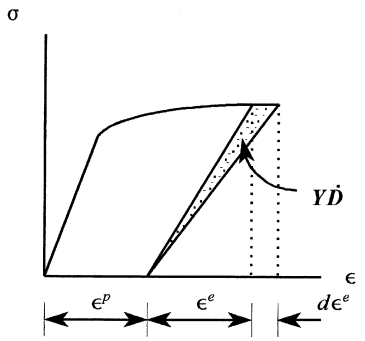

Fig. 4. Damage at constant stress.

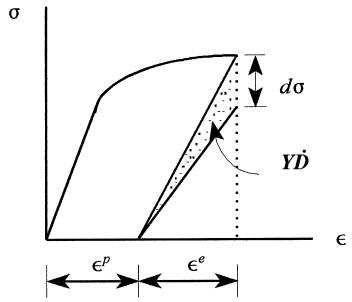

Fig. 5. Damage at constant strain.

8. The damage dissipation inequality

A thermodynamically admissible process (Coleman and Gurtin, 1967; Eringen, 1980) must obey the Second Law of Thermodynamics. Accordingly, we now seek the restrictions that this postulate places on the damage and its evolution.

The local Clausius–Duhem inequality expresses the second law and in the present application has the form

$$\sigma_{ij}\dot{\varepsilon}_{ij} - \rho(\dot{\psi} + \eta\dot{T}) - q_i\frac{T,i}{T} \geqslant 0 \tag{8.1}$$

where η is the specific entropy and $\mathbf{q}$ the heat flux vector.

The Helmholtz free energy is now expressed as

$$\psi = \psi(\boldsymbol{\varepsilon}^e, T, r, \boldsymbol{\alpha}, \mathbf{D}) \tag{8.2}$$

Therefore,

$$\dot{\psi} = \frac{\partial\psi}{\partial\varepsilon^e_{ij}}\dot{\varepsilon}^e_{ij} + \frac{\partial\psi}{\partial T}\dot{T} + \frac{\partial\psi}{\partial r}\dot{r} + \frac{\partial\psi}{\partial\alpha_{ij}}\dot{\alpha}_{ij} + \frac{\partial\psi}{\partial D_{ijkl}}\dot{D}_{ijkl} \tag{8.3}$$

and substitution into (8.1) gives eqn (8.4)

$$\left(\sigma_{ij} - \rho\frac{\partial\psi}{\partial\varepsilon^e_{ij}}\right)\dot{\varepsilon}^e_{ij} + \sigma_{ij}\dot{\varepsilon}^p_{ij} - \rho\left(\frac{\partial\psi}{\partial T} + \eta\right)\dot{T} - \rho\frac{\partial\psi}{\partial r}\dot{r} - \rho\frac{\partial\psi}{\partial\alpha_{ij}}\dot{\alpha}_{ij} - \rho\frac{\partial\psi}{\partial D_{ijkl}}\dot{D}_{ijkl} - q_i\frac{T,i}{T} \geqslant 0 \tag{8.4}$$

Since the entropy is given by

$$\eta = -\frac{\partial\psi}{\partial T} \tag{8.5}$$

eqn (8.4) becomes

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} - R\dot{r} - X^D_{ij}\dot{\alpha}_{ij} + Y_{ijkl}\dot{D}_{ijkl} - q_i\frac{T,i}{T} \geqslant 0 \tag{8.6}$$

If the mechanical and thermal dissipations are not coupled, then

$$\sigma_{ij}\dot{\varepsilon}^{p}_{ij} - R\dot{r} - X^{D}_{ij}\dot{\alpha}_{ij} + Y_{ijkl}\dot{D}_{ijkl} \geqslant 0 \tag{8.7}$$

and

$$-q_i \frac{T,i}{T} \geqslant 0 \tag{8.8}$$

Because plastic flow can occur without damage and vice versa, eqn (8.7) can be further separated into

$$\sigma_{ij}\dot{\varepsilon}^{p}_{ij} - R\dot{r} - X^{D}_{ij}\dot{\alpha}_{ij} \geqslant 0 \tag{8.9}$$

$$Y_{ijkl}\dot{D}_{ijkl} \geqslant 0 \tag{8.10}$$

The first of these is the well known condition on the work in plastic deformation and the second, using (7.12) and (7.13), specifies the condition of irreversibility of damage as the dissipation inequalities

$$\mathrm{d}w_e|_{\sigma} = Y_{ijkl}\,\mathrm{d}D_{ijkl} \geqslant 0 \tag{8.11}$$

$$\mathrm{d}w_e|_{\varepsilon^e} = -Y_{ijkl}\,\mathrm{d}D_{ijkl} \leqslant 0 \tag{8.12}$$

9. The irreversibility condition

In terms of the isotropic damage parameters, the damage rate from eqn (1.3) is

$$\dot{D}_{ijkl} = \dot{D}_2\delta_{ij}\delta_{kl} + \tfrac{1}{2}(\dot{D}_1 - \dot{D}_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{9.1}$$

and the dissipation inequality (8.10) becomes

$$Y_{ijkl}\dot{D}_{ijkl} = \dot{D}_2\,Y_{iijj} + (\dot{D}_1 - \dot{D}_2)\,Y_{ijij} \geqslant 0 \tag{9.2}$$

This is the restriction on any damage path within the damage domain of Fig. 1. This limitation of non-negative damage dissipation can be restated and interpreted more simply for damage paths in Fig. 1. We note first the following:

Using eqn (6.10)

$$Y_{ijij} = \tfrac{1}{2}E_{ijkl}\varepsilon^{e}_{kl}\varepsilon^{e}_{ij} \tag{9.3}$$

or

$$Y_{ijij} = \frac{Ev}{2(1-2v)(1+v)}(\varepsilon^{e}_{ii})^2 + \frac{E}{2(1+v)}(\varepsilon^{e^2}_{11} + \varepsilon^{e^2}_{22} + \varepsilon^{e^2}_{33} + 2\varepsilon^{e^2}_{12} + 2\varepsilon^{e^2}_{13} + 2\varepsilon^{e^2}_{23}) \tag{9.4}$$

and

$$Y_{iijj} = \frac{1}{2}E_{ijkl}\varepsilon^{e}_{kl}\varepsilon^{e}_{ii} = \frac{E}{2(1-2v)}(\varepsilon^{e}_{ii})^2 = \frac{3}{2}K(\varepsilon^{e}_{ii})^2 \tag{9.5}$$

eqn (9.3) is recognized as the strain energy of the virgin material (w^0_e) under the deformation present in the damaged material so that

$$Y_{ijij} = \tfrac{1}{2}\tilde{\sigma}_{ij}\varepsilon^e_{ij} = w^0_e \tag{9.6}$$

Furthermore, in the form of eqn (9.5) is recognized the expression for the strain energy of volumetric deformation in the virgin material (w^0_H) because by definition

$$\tfrac{1}{3}\tilde{\sigma}_{jj} = K\varepsilon^e_{jj} \tag{9.7}$$

and

$$w^0_H = \tfrac{1}{2}(\tfrac{1}{3}\tilde{\sigma}_{jj})\varepsilon^e_{ii} = \tfrac{1}{2}K(\varepsilon^e_{ii})^2 \tag{9.8}$$

Therefore from eqn (9.5)

$$Y_{iijj} = 3w^0_H \tag{9.9}$$

The expression (9.6) can also be rewritten in terms of hydrostatic and distortional (W^0_D) strain energies in the virgin material using the general result

$$w^0_e = w^0_H + w^0_D \tag{9.10}$$

The law of non-negative damage dissipation eqn (9.2) can then be written using eqns (9.6), (9.9) and (9.10) in the form

$$Y_{ijkl}\dot{D}_{ijkl} = w^0_H(\dot{D}_1 + 2\dot{D}_2) + w^0_D(\dot{D}_1 - \dot{D}_2) \geqslant 0 \tag{9.11}$$

When expressed in terms of the physically significant damage parameters relating to bulk and shear responses, one obtains the physically meaningful form of irreversibility of damage

$$Y_{ijkl}\dot{D}_{ijkl} = w^0_H\dot{D}_K + w^0_D\dot{D}_S \geqslant 0 \tag{9.12}$$

The various forms of the condition on damage dissipation, eqns (9.2), (9.11), (9.12), regulate the evolution of damage along any possible path in the damage domain of Fig. 1. We may observe the impact of this limitation on several specific paths previously identified in Fig. 1.

1. On OA defined by $\dot{D}_1 = \dot{D}_2$ (and $\dot{D}_S = 0$) eqn (9.2) shows that the damage can only progress from O to A for which

$$\dot{D}_1 = \dot{D}_2 \geqslant 0 \tag{9.13}$$

We see from eqn (9.12) that under such restrictions, $\dot{D}_K \geqslant 0$ which is the physically recognizable limitation on damage evolution.

2. On AB for which we find $D_2 = v(1-D_1)$ and $\dot{D}_2 = -v\dot{D}_1$, eqn (9.2) becomes

$$Y_{ijkl}\dot{D}_{ijkl} = \dot{D}_1[(1+v)Y_{ijij} - vY_{iijj}] \geqslant 0 \tag{9.14}$$

eqns (9.14), (9.4) and (9.5) show that the damage must proceed with $\dot{D}_1 \geqslant 0$ and $\dot{D}_2 \leqslant 0$ which means that the path can only go from A toward B.

3. On OC for which $D_1 + 2D_2 = 0$, the damage dissipation inequality (9.2) shows that the damage can only progress from O to C with $\dot{D}_1 \geqslant 0$ and $\dot{D}_2 \leqslant 0$.

4. On CB for which $D_2 = -(1-D_1)$ and $\dot{D}_1 = \dot{D}_2$, eqn (9.2) shows that the damage path can only go from C–B with $\dot{D}_1 \geqslant 0$ and $\dot{D}_2 \geqslant 0$.

5. For the so-called $(1-D)$ type of isotropic damage which corresponds to the damage path OB we have $D_2 = 0$. Equation (9.2) shows that the damage must proceed from O to B with $\dot{D}_1 \geqslant 0$.

10. The dissipation potential

The concept of a generalized potential has been introduced by Moreau (1970) and Rice (1970, 1971) to define the rate equations for the internal variables of the dissipation process. The kinetic constitutive equations will describe here the evolution of damage as well as the plastic strains. We shall postulate a potential of dissipation, F, written as a function of the stress variables:

$$F = F(\boldsymbol{\sigma}, R, \mathbf{X^D}, \bar{\mathbf{Y}}) \tag{10.1}$$

According to the hypothesis of generalized normality and the description of rate-independent plasticity (Lubliner, 1990), the evolution or rates of plastic strains and damage are found from F with a positive scalar multiplier $\dot{\lambda}$:

$$\dot{\varepsilon}^P_{ij} = \frac{\partial F}{\partial \sigma_{ij}}\dot{\lambda} \tag{10.2}$$

$$\dot{r} = -\frac{\partial F}{\partial R}\dot{\lambda} \tag{10.3}$$

$$\dot{\alpha}_{ij} = -\frac{\partial F}{\partial X^D_{ij}}\dot{\lambda} \tag{10.4}$$

$$\dot{D}_{ijkl} = -\frac{\partial F}{\partial \bar{Y}_{ijkl}}\dot{\lambda} \tag{10.5}$$

We shall further postulate that the dissipations from damage and plasticity contribute independently to the total potential in the form

$$F = F_P(\boldsymbol{\sigma}, R, \mathbf{X^D}) + F_D(\bar{\mathbf{Y}}) \tag{10.6}$$

where F_P is a plastic potential and F_D a damage potential dependent on the thermodynamic conjugate of the damage alone. Then, using eqns (10.2)–(10.5), the damage evolution law will be determined by the selection of F_D just as the kinetic law of plastic flow comes from F_P.

If σ_y is the yield stress of the material in its virgin state, yielding in the damaged material will occur under uniaxial stress when

$$\tilde{\sigma} = \sigma_y \tag{10.7}$$

This might be generalized to three dimensions using the equivalent stress concept for the effective stress

$$\tilde{\sigma}_{eq} = \left(\tfrac{3}{2}\tilde{\sigma}^D_{ij}\tilde{\sigma}^D_{ij}\right)^{1/2} = \sigma_y \tag{10.8}$$

One may account for kinematic and isotropic hardening by expressing the yield function for such a material in the form

$$f = (\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq} - R - \sigma_y \tag{10.9}$$

where

$$(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq} = [\tfrac{3}{2}(\tilde{\sigma}^D_{ij} - X^D_{ij})(\tilde{\sigma}^D_{ij} - X^D_{ij})]^{1/2} \tag{10.10}$$

In the case of an associated flow rule for the plastic deformations, f in eqn (10.9) is taken as the potential function F_P, and by virtue of the effective stress tensor $\tilde{\boldsymbol{\sigma}}$ used in this potential, it will include also plastic deformation induced by the damage. Non-associated flow rules could also be considered in this way by taking a plastic potential F_P other than the yield function f (e.g. Lemaître and Chaboche, 1990)

$$F_P = f + \frac{3}{4X_\infty} X^D_{ij} X^D_{ij} \tag{10.11}$$

where the additional expression accounts for non-linear kinematic hardening and X_∞ is a material parameter. We shall use this form in the subsequent derivations.

If the dissipation potential for the damage is written as a function of $\mathbf{Y}$ rather than $\bar{\mathbf{Y}}$, eqns (10.6) and (10.5) give the damage evolution equation

$$\dot{D}_{ijkl} = -\frac{\partial F}{\partial \bar{Y}_{ijkl}} \dot{\lambda} = \frac{\partial F_D}{\partial Y_{ijkl}} \dot{\lambda} \tag{10.12}$$

The potential function F_D cannot be chosen arbitrarily, however, We note for example that eqn (9.1), with $i \neq j$, requires that

$$\dot{D}_{(ijij)} = \tfrac{1}{2}(\dot{D}_1 - \dot{D}_2) = \tfrac{1}{2}(\dot{D}_{(iiii)} - \dot{D}_{(iijj)}) \tag{10.13}$$

with no summation on indices within parentheses. Therefore, we see that F_D must be such that

$$\frac{\partial F_D}{\partial Y_{(ijij)}} = \frac{1}{2}\left(\frac{\partial F_D}{\partial Y_{(iiii)}} - \frac{\partial F_D}{\partial Y_{(iijj)}}\right) \tag{10.14}$$

Aside from satisfying this constraint, the potential F_D is chosen to reproduce observed damage evolution. Several simple choices of F_D are explored in the following.

1. One parameter damage

If S is a material constant, then

$$F_D(\mathbf{Y}) = \frac{Y_{mnmn} Y_{pqqp}}{2S} \tag{10.15}$$

gives a damage evolution rule

$$\dot{D}_{ijkl} = \frac{Y_{mnmn}}{2S}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\dot{\lambda} \tag{10.16}$$

which leads to the following:

$$\dot{D}_{(iiii)} = \dot{D}_1 = \frac{Y_{mnmn}}{S}\dot{\lambda} = \frac{w^0_e}{S}\dot{\lambda} \tag{10.17}$$

$$\dot{D}_{(iijj)} = \dot{D}_2 = 0 \tag{10.18}$$

$$\dot{D}_{(ijij)} = \frac{1}{2}\dot{D}_1 = \frac{Y_{mnmn}}{2S}\dot{\lambda} = \frac{w_e^0}{2S}\dot{\lambda} \tag{10.19}$$

In other words, this choice of F_D gives $\dot{D}_2 = 0$ and would correspond to the so-called $(1-D)$ type of isotropic damage, i.e. the one parameter description of damage. This is damage path OB in Fig. 1.

2. For the potential

$$F_D(\mathbf{Y}) = \frac{Y_{mmnn}Y_{ppqq}}{2S} \tag{10.20}$$

eqn (10.12) gives

$$\dot{D}_{ijkl} = \frac{Y_{mmnn}}{S}\delta_{ij}\delta_{kl}\dot{\lambda} \tag{10.21}$$

from which we find

$$\dot{D}_{(iiii)} = \dot{D}_1 = \frac{Y_{mmnn}}{S}\dot{\lambda} = \frac{3w_H^0}{S}\dot{\lambda} \tag{10.22}$$

$$\dot{D}_{(iijj)} = \dot{D}_2 = \frac{3w_H^0}{S}\dot{\lambda} \tag{10.23}$$

$$\dot{D}_{(ijij)} = 0 \tag{10.24}$$

This F_D corresponds to damage evolution along path OA in Fig. 1 with $\dot{D}_1 = \dot{D}_2$. This potential appears to be rather restrictive because it indicates that damage will only occur if there is non zero volumetric deformation. Indeed, in the case where $w_H^0 = 0$, i.e. $\tilde{\boldsymbol{\sigma}}_{kk} = 0$, we would always have $\dot{D}_1 = \dot{D}_2 = 0$ which would imply no damage under a purely distortional deformation.

3. Selection of F_D in the form

$$F_D(\mathbf{Y}) = \frac{Y_{ppqq}Y_{mnmn}}{S} \tag{10.25}$$

gives

$$\dot{D}_{ijkl} = \frac{1}{2S}[Y_{ppqq}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + 2Y_{mnmn}\delta_{ij}\delta_{kl}]\dot{\lambda} \tag{10.26}$$

from which

$$\dot{D}_{(iiii)} = \dot{D}_1 = \frac{Y_{ppqq}+Y_{mnmn}}{S}\dot{\lambda} = \frac{3w_H^0+w_e^0}{S}\dot{\lambda} \tag{10.27}$$

$$\dot{D}_{(iijj)} = \dot{D}_2 = \frac{Y_{mnmn}}{S}\dot{\lambda} = \frac{w_e^0}{S}\dot{\lambda} \tag{10.28}$$

$$\dot{D}_{(ijij)} = \frac{1}{2}(\dot{D}_1 - \dot{D}_2) = \frac{3w_H^0}{2S}\dot{\lambda} \tag{10.29}$$

In this case, and in the potentials F_D explored above, the damage variables D_1 and D_2 always increase if S is positive. However, we have already seen that the dissipation inequality, eqn (9.4), allows a decreasing D_2 within the damage domain.

Although the best choice of the dissipation potential for a given material remains to be determined and certainly must depend on the type of the material, we propose a function that can satisfy the general restrictions of isotropy and still permit the latitude given by the dissipation inequality:

$$F_D = \frac{1}{2S}(Y_{mnmn}Y_{pqpq} \pm \tfrac{1}{3}Y_{mmnn}Y_{ppqq}) \tag{10.30}$$

The damage flow rule becomes

$$\dot{D}_{ijkl} = \left[\frac{w_e^0}{2S}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \pm \frac{w_H^0}{S}\delta_{ij}\delta_{kl}\right]\dot{\lambda} \tag{10.31}$$

which gives individual damage rates

$$\dot{D}_{(iiii)} = \dot{D}_1 = \frac{w_e^0 \pm w_H^0}{S}\dot{\lambda} \tag{10.32}$$

$$\dot{D}_{(iijj)} = \dot{D}_2 = \pm\frac{w_H^0}{S}\dot{\lambda} \tag{10.33}$$

$$\dot{D}_{(ijij)} = \tfrac{1}{2}(\dot{D}_1 - \dot{D}_2) = \frac{w_e^0}{2S}\dot{\lambda} \tag{10.34}$$

11. The kinetic constitutive equations

Using the potentials outlined in the preceding for F_D and F_P in the generalized potential F, eqn (10.6), we can develop the constitutive equations for an elastic/plastic material with isotropic damage. This is accomplished by evaluating the four rates of eqns (10.2)–(10.5) using these potentials. The damage rate of eqn (10.5) is already evaluated from the potential F_D as given by eqn (10.31). We proceed next to evaluate the plastic strain rate in eqn (10.2).

We note first that the deviatoric effective stress

$$\tilde{\sigma}_{ij}^D = \tilde{\sigma}_{ij} - \tfrac{1}{3}\tilde{\sigma}_{kk}\delta_{ij} \tag{11.1}$$

can be written using eqn (5.3) as

$$\tilde{\sigma}_{ij}^D = (\Lambda_{ijmn} - \tfrac{1}{3}\Lambda_{kkmn}\delta_{ij})\sigma_{mn} \tag{11.2}$$

Using eqn (10.11) in eqns (10.6) and (10.2), the plastic strain rate tensor for the damaged material is

$$\dot{\varepsilon}^P_{rs} = \frac{\partial F_P}{\partial \sigma_{rs}}\dot{\lambda} = \frac{\partial f}{\partial \sigma_{rs}}\dot{\lambda} = \frac{\partial}{\partial \sigma_{rs}}(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}\dot{\lambda} \tag{11.3}$$

From eqn (10.10) we find

$$\frac{\partial}{\partial \sigma_{rs}}(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq} = \frac{3}{2}\frac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\frac{\partial \tilde{\sigma}^D_{ij}}{\partial \sigma_{rs}} \tag{11.4}$$

and from (11.2)

$$\frac{\partial \tilde{\sigma}^D_{ij}}{\partial \sigma_{rs}} = \Lambda_{ijrs} - \frac{1}{3}\Lambda_{kkrs}\delta_{ij} \tag{11.5}$$

These latter two equations substituted back into eqn (11.3) give the kinetic equation for the plastic strain rate

$$\dot{\varepsilon}^P_{rs} = \frac{3}{2}\frac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\Lambda_{ijrs}\dot{\lambda} \tag{11.6}$$

It is common to compute an equivalent plastic strain rate, often called the accumulated plastic strain rate $\dot{p}$ in the damaged material by the definition

$$\dot{p} = (\tfrac{2}{3}\dot{\varepsilon}^P_{ij}\dot{\varepsilon}^P_{ij})^{1/2} \tag{11.7}$$

This can be written in terms of the effective stresses upon substitution into eqn (11.6) to give

$$\dot{p} = \frac{1}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\left[\frac{3}{2}(\tilde{\sigma}^D_{ij} - X^D_{ij})(\tilde{\sigma}^D_{kl} - X^D_{kl})\Lambda_{ijrs}\Lambda_{klrs}\right]^{1/2}\dot{\lambda} \tag{11.8}$$

A much simpler form can be obtained by using eqn (5.2) and eqn (10.10) in eqn (11.8) to express $\dot{p}$ in terms of the parameters D_1 and D_2 of isotropic damage or the shear damage parameter D_S:

$$\dot{p} = \frac{1}{(1 - D_1 + D_2)}\dot{\lambda} = \frac{1}{(1 - D_S)}\dot{\lambda} \tag{11.9}$$

The plastic strain rate tensor in the undamaged material when it is deformed to the same elastic strain tensor is found from eqn (10.11)

$$\dot{\varepsilon}^{P_0}_{rs} = \frac{\partial F_P}{\partial \tilde{\sigma}_{rs}}\dot{\lambda} = \frac{\partial f}{\partial \tilde{\sigma}_{rs}}\dot{\lambda} = \frac{\partial f}{\partial \sigma_{ij}}\frac{\partial \sigma_{ij}}{\partial \tilde{\sigma}_{rs}}\dot{\lambda} = \frac{3}{2}\frac{(\tilde{\sigma}^D_{rs} - X^D_{rs})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\dot{\lambda} \tag{11.10}$$

From this and using eqn (11.9), it is seen that the plastic strain rates in the damaged material and the undamaged material are related simply by

$$\dot{\varepsilon}^P_{rs} = \dot{\varepsilon}^{P_0}_{ij}\Lambda_{ijrs} \tag{11.11}$$

or conversely,

$$\dot{\varepsilon}^{P_0}_{rs} = \dot{\varepsilon}^P_{ij}\Lambda^{-1}_{ijrs} = \dot{\varepsilon}^P_{ij}(I_{ijrs} - D_{ijrs}) \tag{11.12}$$

The accumulated plastic strain rate $\dot{r}$ in the undamaged material was defined by eqn (6.12) and

is the counterpart of $\dot{p}$ for the undamaged material. It is found as the second kinetic equation, eqn (10.3). Using eqns (10.6) and (10.11)

$$\dot{r} = \dot{\lambda} \tag{11.13}$$

This same result would be obtained if the strain rates eqn (11.10) are substituted into the definition of $\dot{r}$, eqn (6.12).

For the so-called $(1-D)$ type of isotropic damage for which

$$\Lambda_{ijrs} = \frac{1}{(1-D)} I_{ijrs} \tag{11.14}$$

we find from eqn (11.8) and eqn (11.13) that

$$\dot{r} = \dot{p}(1-D) \tag{11.15}$$

When there is no damage, $\mathbf{D} = \mathbf{0}$, the damage operator $\mathbf{\Lambda}$ reduces to the identity tensor $\mathbf{I}$, and we find from eqns (11.8) and (11.13)

$$\dot{p} = \dot{r} = \dot{\lambda} \quad \text{for } \mathbf{D} = \mathbf{0} \tag{11.16}$$

The same is seen for the $(1-D)$ description, eqn (11.15).

The third kinetic equation, eqn (10.4), gives the back strain rate tensor for the undamaged material using eqns (10.6) and (10.11):

$$\dot{\alpha}_{rs} = -\frac{\partial F_P}{\partial X^D_{rs}}\dot{\lambda} = -\left[\frac{\partial f}{\partial X^D_{rs}} + \frac{3}{2X_\infty} X^D_{rs}\right]\dot{\lambda} \tag{11.17}$$

From eqn (10.9),

$$\frac{\partial f}{\partial X^D_{rs}} = -\frac{3}{2}\frac{(\tilde{\sigma}^D_{rs} - X^D_{rs})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}} \tag{11.18}$$

which together with eqn (11.12) can be substituted into eqn (11.17) to give the back strain rate tensor

$$\dot{\alpha}_{rs} = \dot{\varepsilon}^p_{ij}\Lambda^{-1}_{ijrs} - \frac{3}{2X_\infty} X^D_{rs}\dot{\lambda} \tag{11.19}$$

or

$$\dot{\alpha}_{rs} = \dot{\varepsilon}^{p_0}_{rs} - \frac{3}{2X_\infty} X^D_{rs}\dot{\lambda} \tag{11.20}$$

12. The stress variables X^D and R

We have obtained in the preceding the four kinetic constitutive eqns (10.31), (11.6), (11.13), (11.20) from the selected potential functions F_D and F_P of eqns (10.11) and (10.30). In these flow

rules, the back stress tensor $\mathbf{X^D}$ and the isotropic hardening variable R for the undamaged material remain to be determined.

The back stress tensor depends on the plastic properties of the material. A relation often used is

$$X^D_{rs} = \tfrac{2}{3} X_\infty \gamma \alpha_{rs} \tag{12.1}$$

where γ is a characteristic coefficient of the material (Lemaître and Chaboche, 1990). For such a choice, eqn (11.19) gives a value of the back stress rate tensor

$$\dot{X}^D_{rs} = \gamma[\tfrac{2}{3} X_\infty \dot{\varepsilon}^P_{ij} \Lambda^{-1}_{ijrs} - X^D_{rs}\dot{\lambda}] \tag{12.2}$$

which can also be written, using eqn (11.12),

$$\dot{X}^D_{rs} = \gamma[\tfrac{2}{3} X_\infty \dot{\varepsilon}^{P_0}_{rs} - X^D_{rs}\dot{\lambda}] \tag{12.3}$$

This last equation, familiar in plasticity, is a modification of the Melan–Prager model with an additional term on the right-hand side that represents 'fading strain memory' (Il'iushin, 1954), and allows a better description of the Bauschinger effect (Lubliner, 1990).

Substituting eqn (11.10) into eqn (12.3), the rate equation of the back stress tensor in terms of the effective stresses is

$$\dot{X}^D_{rs} = \gamma \left[X_\infty \frac{(\tilde{\sigma}^D_{rs} - X^D_{rs})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X^D})_{eq}} - X^D_{rs} \right] \dot{\lambda} \tag{12.4}$$

The isotropic hardening R, eqn (6.13), which is identified as the thermodynamic variable corresponding to the accumulated plastic strain r for the undamaged material, increases non-linearly with the plastic strain and tends to some limiting value R_∞, as observed in various cyclic tests (Amar and Dufailly, 1993). It is often taken in the form

$$R = R_\infty[1 - \exp(-br)] \tag{12.5}$$

where b is a material parameter. For such a choice, the rate equation, using eqn (11.13), becomes

$$\dot{R} = b(R_\infty - R)\dot{\lambda} \tag{12.6}$$

13. The plastic multiplier

The multiplier $\dot{\lambda}$ is found by requiring consistency with the yield condition. The plastic flow occurs, as given by eqn (11.3) and eqn (11.10), when the material is at yield and remains at yield, i.e. $f = 0$ and $\dot{f} = 0$. Therefore, using eqn (10.9)

$$\dot{f} = \frac{\partial f}{\partial \sigma_{rs}} \dot{\sigma}_{rs} + \frac{\partial f}{\partial X^D_{rs}} \dot{X}^D_{rs} + \frac{\partial f}{\partial D_{pqrs}} \dot{D}_{pqrs} + \frac{\partial f}{\partial R} \dot{R} = 0 \tag{13.1}$$

where it can be shown that

$$\frac{\partial f}{\partial D_{pqrs}} = \frac{3}{4}\frac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}(\Lambda_{ijpq}\tilde{\sigma}^D_{rs} + \Lambda_{ijrs}\tilde{\sigma}^D_{pq}) \tag{13.2}$$

Using eqns (11.4), (11.18), (13.2) in eqn (13.1) and noting that $(\partial f/\partial R) = -1$ we find: eqn (13.3)

$$\frac{3}{2}\frac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\Lambda_{ijrs}\dot{\sigma}_{rs} - \frac{3}{2}\frac{(\tilde{\sigma}^D_{rs} - X^D_{rs})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\dot{X}^D_{rs} + \frac{3}{4}\frac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}(\Lambda_{ijpq}\tilde{\sigma}^D_{rs} + \Lambda_{ijrs}\tilde{\sigma}^D_{pq})\dot{D}_{pqrs} - \dot{R} = 0 \tag{13.3}$$

Finally, substituting eqn (12.4) and eqn (12.6) into eqn (13.3), we obtain

$$\dot{\lambda} = \frac{\dfrac{3}{2}\dfrac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\Lambda_{ijrs}\dot{\sigma}_{rs}}{\gamma X_\infty + b(R_\infty - R) - \dfrac{3}{2}\dfrac{(\tilde{\sigma}^D_{ij} - X^D_{ij})}{(\tilde{\boldsymbol{\sigma}}^{\mathbf{D}} - \mathbf{X}^{\mathbf{D}})_{eq}}\left[\gamma X^D_{ij} + \dfrac{1}{2}(\Lambda_{ijpq}\tilde{\sigma}^D_{rs} + \Lambda_{ijrs}\tilde{\sigma}^D_{pq})\dfrac{\partial F_D}{\partial Y_{pqrs}}\right]} \tag{13.4}$$

The basic variables and equations that are used in a solution for an elastoplastic material, initially isotropic and sustaining isotropic damage, are: (a) the damage operator Λ, eqn (5.2); (b) the strain tensor $\boldsymbol{\varepsilon}$, eqn (6.2); (c) the elastic strain tensor $\boldsymbol{\varepsilon}^e$, eqn (6.9); (d) the plastic strain increment $\dot{\varepsilon}^p$, eqn (11.6); (e) the accumulated plastic strain rate $\dot{p}$, eqn (11.8); (f) the back stress rate tensor $\dot{\mathbf{X}}^{\mathbf{D}}$, eqn (12.4); (g) the isotropic hardening rate $\dot{R}$, eqn (12.6); (h) the plastic multiplier $\dot{\lambda}$, eqn (13.4); (i) and the damage evolution equations, eqn (10.31). Five material parameters are needed: S in eqn (10.31); X_∞ and γ in eqn (12.4); R_∞ and b in eqn (12.6).

The detailed equations given in the preceding development are limited by the specific choices of the following assumed forms:

(a) the dissipation potential that defines damage evolution, eqn (10.30)
(b) the plastic potential (Mises), eqns (10.9) and (10.11)
(c) the back stress tensor, eqn (12.1)
(d) the isotropic hardening parameter, eqn (12.5).

14. Conclusion

The case of isotropic damage that takes an isotropic undamaged material to an isotropic damaged state is characterized by two independent damage parameters. Damage can proceed along many possible paths within a limited region of the space of those damage parameters that we have called the damage domain. The actual path and the rules governing the evolution of damage are given by the material constitutive equations including damage effects.

In this work we have given the appropriate equations for an elastoplastic material with a specific type of damage behavior based on an assumed damage potential. These governing equations have as their unifying base the thermodynamics of irreversible processes.

The present work, however, has not treated the significant question of unequal responses in tension and compression. This may have special relevance in cases of damage related to microcracking and will need to be included in subsequent extension of this work.

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